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# RADAR CFAR THRESHOLDING IN CLUTTER UNDER DETECTION OF AIRBORNE BIRDS

N.A. Nechval

Department of Control Systems
Aviation University of Riga
1, Lomonosov Street
226019 Riga
Latvia

## ABSTRACT

The study of radar returns from birds is an interesting but rather neglected area of practical importance to radar operations, the hazard of collisions between birds and aircraft, and environmental monitoring, as well as to the subject of ornithology itself. The abundant radar returns from birds should not always be regarded merely as spurious signals to be eliminated from surveillance radar displays, but as useful signals to be made available for their utility in combating bird hazards and in environmental monitoring. This view, however, does not seem to be widely accepted as yet by radar engineers. Tests have shown that radar returns from birds can be readily distinguished from radar returns from automobiles and aircraft. With experience, an observer can distinguish large birds from small birds and perhaps make more subtle distinctions. This paper is concerned with the problem of statistical classification of radar clutter into one of several categories, including airborne birds, bats, insects, weather, and target classes, as well as the corrupting background noise. The problem can be separated into two parts. The first part, the problem focused on here, is to decide whether the received signal is a signal plus noise (corresponding to the presence of any of various types of radar clutter corrupted by the background noise) or noise alone (corresponding to the presence of radar clutter including the corrupting background noise only). Noise is assumed to be Gaussian, but whose covariance matrix is totally unknown. The second part is the discrimination between various types of radar clutter corrupted by the background noise, i.e. classification of radar clutter into one of several categories. A solution to this problem has been presented in Nechval (1991) and will not be considered further. Radar detection procedures involve the comparison of the received signal amplitude to a threshold. The technique presented in this paper allows one to find a detection threshold that achieves a constant false-alarm rate (CFAR) in the presence of intensity changes in the noise background.

## 1. INTRODUCTION

The task of primary radars used in air traffic control is to detect all objects within the area of observation and to estimate their positional coordinates. Generally speaking, target detection would be an easy task if the echoing objects were located in front of an otherwise clear or empty background. In such a case the echo signal can simply be compared with a fixed threshold, and targets are detected whenever the signal exceeds this threshold. In real radar application, however, the target practically always appears before a background filled (mostly in a complicated manner) with point, area, or extended clutter. Frequently the location of this background clutter is additionally subject to variations in time and position. This fact calls for adaptive signal processing techniques operating with a variable detection threshold to be determined in accordance to the local clutter situation. In order to obtain the needed local clutter information, a certain environment defined by a window around the radar test cell must be analyzed.

Usually the background reflectors, undesired as they are from the standpoint of detection and tracking, are denoted by the term "clutter", and in the design of the signal processing circuits the assumption is made that this clutter is uniformly distributed over the entire environment. Signal processing is designed so that, whenever possible, target reports are received from useful targets only, rather than from background reflectors.

In practice, however, clutter phenomena may be caused by a number of different sources (such as airborne birds, bats, insects, or small clouds or other meteorological structures). Improvements in target detection and clutter suppression over the present state of the art can be effected only by removing the simplifying assumptions step by step and introducing a more differentiating way of argumentation. Ultimately it may become necessary to identify clutter regions of differing clutter type and to describe their properties such as type, size and borders, power, and spectral features rather than trying to suppress and ignore them at an early stage of signal processing. Thus for discriminating targets (such as aircraft) from clutter, it might be useful to build up a complete "image" of the clutter situation encountered in the overall observation space.

Unfortunately, real-time information on airborne hazards to aircraft, such as birds and storm systems, is also suppressed. The ability to classify clutter and hence identify these hazards can thus contribute significantly to air traffic safety.

These ideas reflect a trend presently observed in radar signal processing philosophy, a trend to regard the problem of target detection and clutter suppression more and more as a problem of image processing and image analysis. The procedure outlined in the following is one of steps in this direction.

In a radar system, the detection of signals in a background of stationary noise usually involves the comparison of a statistic, based on samples of signal plus noise, with a constant threshold that is determined from the noise-only probability distribution. The threshold is chosen so that a specified false-alarm probability is achieved, using the so-called Neyman-Pearson criterion. Unfortunately, in most cases the threshold depends on parameters of the noise-only distribution and, in practice, these parameters are actually unknown. If these parameters can be estimated from the available data, then the threshold can be determined (at least approximately) from the estimated parameters.

In a nonhomogeneous noise environment in which the average noise power varies in an unknown manner, it is impossible to maintain a constant false-alarm rate using a fixed detection threshold. In order to maintain a constant false-alarm rate (CFAR), it is necessary to utilize an adaptive threshold which can adjust to varying noise levels.

This paper is concerned with the problem of statistical classification of radar clutter into one of several categories, including airborne birds, bats, insects, weather, and target classes, as well as the corrupting background noise. The problem can be separated into two parts. The first part, the problem focused on here, is to decide whether the received signal is a signal plus noise (corresponding to the presence of any of various types of radar clutter corrupted by the background noise) or noise alone (corresponding to the presence of radar clutter including the corrupting background noise only). Noise is assumed to be Gaussian, but whose covariance matrix is totally unknown. The second part is the discrimination between various types of radar clutter corrupted by the background noise, i.e. classification of radar clutter into one of several categories, including airborne birds, bats, insects, weather, and target classes, if the received signal is a signal plus noise. The process of classification can be formalized as follows. The unprocessed radar data is passed through a feature extractor, which transforms the available data samples into a set of separable features. These features are derived from the reflection coefficients computed using the multisegment version of Burg's formula (Kay and Makhoul, 1983; Stehwien and Haykin, 1989). The aforementioned coefficients (that contain all spectral information, including the mean doppler shift) are then transformed and grouped to satisfy the requirements for multivariate Gaussian behaviour. Only information which is different from class to class is maintained, and in such a form that a reliable decision, based on a discriminant function derived from the above features, may be made. A solution to this problem has been presented in Nechval (1991) and will not be considered further.

The purpose of this paper is to present the procedure which allows one to find a detection threshold that achieves a fixed probability of a false alarm which is invariant to intensity changes in the noise background.

## 2. PROBLEM FORMULATION

Let $Z = (Z(1), Z(2), \dots, Z(n))$ be a random sample formed of n independent and identically distributed clutter observations. On the basis of these observations we are to decide which of the following hypotheses is true:

$H_0$ (the noise-alone hypothesis):

$$Z(i) = Z^\circ(i) = (Z_1^\circ(i), \dots, Z_p^\circ(i))' \sim N_p(0,Q), \quad i=1(1)n, \tag{1}$$

where Q is the unknown covariance matrix;

$H_1$ (the signal-plus-noise hypothesis):

$$Z(i) = Z^\circ(i) + bS(i), \quad i=1(1)n, \tag{2}$$

where $Z^\circ(i)$ is the vector representing all sources of noise-only processes,

$$S = (S(1), \dots, S(n)) \tag{3}$$

is the signal pattern (an n-element row vector), and

$$b = (b_1, \dots, b_p)' \tag{4}$$

is a (p x 1) vector of unknown signal intensities corresponding to the p features of the signal, respectively.

## 3. THE MAXIMUM LIKELIHOOD RATIO TEST

In order to distinguish the two hypotheses ($H_0$ and $H_1$) the maximum likelihood ratio testing procedure is used, where the probability density function of the sample data is maximized over all unknown parameters, separately for each of the two hypotheses. The ratio of these maxima is the detection statistic, and the hypothesis whose probability density function is in the numerator is accepted as true if it exceeds some preassigned threshold. The maximizing parameter values are, by definition, the maximum likelihood estimators of these parameters, hence the maximized probability functions are obtained by replacing the unknown parameters by their maximum likelihood estimators.

The maximum likelihood ratio principle is best described by a likelihood ratio defined on some sample space $\ddot{Z}$ with a parameter set $\ddot{\theta}$. Since

$$Q = E((Z(i)-E(Z(i)))(Z(i)-E(Z(i)))') \tag{5}$$

is the unknown covariance matrix of random vector Z(i) for i=1, 2, ... ,n, then for the current problem

$$\ddot{\theta} \triangleq \{\theta=(b,Q): Q>0\} \tag{6}$$

and the likelihood function is

$$L(Z;\theta) = \frac{1}{2\pi^{np/2}\,|Q|^{n/2}} \cdot \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(Z(i)-E(Z(i)))'Q^{-1}(Z(i)-E(Z(i)))\right] =$$

$$= \frac{1}{2\pi^{np/2}\,|Q|^{n/2}} \exp\left[-\frac{1}{2}\mathrm{Tr}(Q^{-1}(Z-E(Z))(Z-E(Z))')\right], \tag{7}$$

where $|Q| \neq 0$ is the determinant of $Q$ and

$$Z = (Z(1), \ldots, Z(n)) \tag{8}$$

is a $p \times n$ matrix of vector data $Z(i)$ and Tr denotes the matrix trace function.

Let $\ddot{w}^\circ$ be the region in the parameter space $\ddot{\theta}$ specified by the $H_0$ hypothesis, then in terms of the complementary subsets $\ddot{w}^\circ$ and $\ddot{\theta}-\ddot{w}^\circ$ of $\ddot{\theta}$ define the alternative hypotheses $H_0$ and $H_1$ as follows:

$$H_0 \equiv (b,Q)\in \ddot{w}^\circ \quad \text{and} \quad H_1 \equiv (b,Q)\in \ddot{\theta} - \ddot{w}^\circ, \tag{9}$$

where

$$\ddot{w}^\circ = \left\{ (0,Q):Q>0 \right\}, \tag{10}$$

$$\ddot{\theta} - \ddot{w}^\circ = \left\{ (b,Q):Q>0,\ b\neq 0 \right\}. \tag{11}$$

The maximum likelihood ratio test is given by

$$LR(Z) = \frac{\max\limits_{\theta\in\ddot{\theta}-\ddot{w}^\circ} L(Z;\theta)}{\max\limits_{\theta\in\ddot{w}^\circ} L(Z;\theta)} \quad \begin{matrix} \geq c, & \text{then } H_1 \\ < c, & \text{then } H_0 \end{matrix} \tag{12}$$

where $c\geq 0$ is the threshold of test.

It can be shown that

$$\max_{\theta\in\ddot{\theta}-\ddot{w}^\circ} L(Z;\theta) = \frac{1}{2\pi^{np/2}\,|\hat{Q}_b|^{n/2}} \exp(-np/2) \tag{13}$$

and

$$\max_{\theta\in\ddot{w}^\circ} L(Z;\theta) = \frac{1}{2\pi^{np/2}\,|\hat{Q}_0|^{n/2}} \exp(-np/2), \tag{14}$$

where

$$\hat{Q}_0 = \sum_{i=1}^{n} Z(i)Z'(i)/n = (1/n)ZZ', \tag{15}$$

$$\hat{Q}_b = \sum_{i=1}^{n} (Z(i)-\hat{b}S(i))(Z(i)-\hat{b}S(i))'/n$$

$$= (1/n)(Z-\hat{b}S)(Z-\hat{b}S)', \tag{16}$$

and

$$\hat{b} = \frac{ZS'}{SS'} \tag{17}$$

are the well-known maximum likelihood estimators of the unknown parameters Q and b under the hypotheses $H_0$ and $H_1$, respectively. Thus a substitution of (13) and (14) into (12) yields the maximum likelihood ratio test

$$LR(Z) = \frac{|\hat{Q}_0|^{n/2}}{|\hat{Q}_b|^{n/2}} \begin{array}{ll} \geqslant c, & \text{then } H_1 \\ < c, & \text{then } H_0 . \end{array} \tag{18}$$

Taking the n/2th root, this test is evidently equivalent to

$$\dot{LR}(Z) = \frac{|\hat{Q}_0|}{|\hat{Q}_b|} \begin{array}{ll} \geqslant k, & \text{then } H_1 \\ < k, & \text{then } H_0 \end{array} \tag{19}$$

where $k=c^{2/n}$. A substitution of (15), (16), and (17) into (19) produces the explicit test

$$\dot{LR}(Z) = \frac{|ZZ'|}{\left| ZZ' - \frac{(ZS')(ZS')'}{SS'} \right|} \begin{array}{ll} \geqslant k, & \text{then } H_1 \\ < k, & \text{then } H_0 . \end{array} \tag{20}$$

To further simplify (20) note first that the inverse of $ZZ'$ is shown to exist with probability one in the last paragraph of Appendix. Thus the test ratio in (20) can be considerably simplified by factoring out the determinant of the p x p matrix $ZZ'$ in the denominator to obtain this ratio in the form

$$\dot{LR}(Z) = \frac{|ZZ'|}{|ZZ'| \left| I - (ZZ')^{-1/2} \frac{(ZS')(ZS')'}{SS'} (ZZ')^{-1/2} \right|} = \frac{1}{1 - \frac{(ZS')'(ZZ')^{-1}(ZS')}{SS'}} . \tag{21}$$

The last equation follows from a well-known determinant identity. Clearly the test in (21) is equivalent finally to the test

$$W(Z) = \frac{(ZS')'(ZZ')^{-1}(ZS')}{SS'} \begin{array}{ll} \geqslant w_0, & \text{then } H_1 \\ < w_0, & \text{then } H_0 . \end{array} \tag{22}$$

The result (22) is a CFAR test for a signal with unknown relative intensities corresponding to the p features, respectively. The test under the assumption of unknown background clutter statistics has the CFAR property that the probability of a false alarm or its equivalent, the probability of signal detection, given the null or noise-only hypothesis $H_0$, is independent of the actual covariance matrix of the data. If p=1, the resulting test reduces to the standard normalized matched filter test for finding a signal in clutter of unknown and varying intensity.

## 4. DETECTION AND FALSE ALARM PROBABILITIES OF TEST

In order to find the probability density function of the test W in (22) on both hypotheses $H_0$ and $H_1$, one assumes the b and Q are known and start by noting that

$$\mathrm{cov}(Z(i);H_l) = Q \qquad \text{for } l=0,1. \tag{23}$$

Also

$$E(Z(i);H_0) = E(Z^\circ(i)) = 0 \tag{24}$$

and

$$E(Z(i);H_1) = E(Z^\circ(i)+bS(i)) = bS(i) \tag{25}$$

or in terms of Z

$$E(Z;H_0) = 0 \quad \text{and} \quad E(Z;H_1) = bS. \tag{26}$$

Next perform a whitening procedure on Z(i) by defining

$$Y(i) = Q^{-1/2}Z(i), \text{ for } i=1,2, \ldots ,n, \tag{27}$$

i.e., let

$$Y = (Y(1), \ldots ,Y(n)) = Q^{-1/2}Z. \tag{28}$$

The whitening procedure in (27) and the assumption that $Z_j(i)$, i=1(1)n, j=1(1)p, are mutually independent produce the result

$$\mathrm{cov}(Y_j(i)Y_s(r)) = d(j,s)d(i,r) \tag{29}$$

for j,s=1,2, ... ,p and i,r=1,2, ... ,n. Here d(i,r) is the Kronecker delta function defined by

$$d(i,r) = \begin{cases} 1 & \text{if } i=r \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

and $Y_j(i)$ is the jth element of vector Y(i). Then by (26) to (30)

$$E(Y;H_0) = 0, \tag{31}$$

$$E(Y;H_1) = Q^{-1/2}bS, \tag{32}$$

and

$$\mathrm{cov}(Y(i);H_l) = I_p \qquad \text{for } l=0,1, \tag{33}$$

where $I_p$ is the p x p identity matrix.

Evidently by the transformation in (28) the test function in (22) becomes

$$W = \frac{(YS')'(YY')^{-1}(YS')}{SS'} \begin{matrix} \geq w_0, & \text{then } H_1 \\ < w_0, & \text{then } H_0. \end{matrix} \tag{34}$$

Since $SS'$ is a positive scalar, at this point it simplifies matters to normalize the signal vector S by letting

$$S_1 = \frac{S}{(SS')^{1/2}}. \tag{35}$$

Then the test function in (34) becomes, using (35),

$$W = (YS_1')'(YY')^{-1}(YS_1'). \tag{36}$$

By (35) the sum-of-squares norm of $S_1$ is given by $\|S_1\| = 1$. Hence, $S_1$ is a unit row vector in the "direction" of vector S.

Now consider the n x n orthonormal matrix

$$U = \begin{pmatrix} S_1 \\ M \end{pmatrix} \tag{37}$$

where M is a (n-1) x n matrix, composed of some set of orthonormal row vectors, and such that

$$S_1M' = 0. \tag{38}$$

Hence the matrix U carries out rotations in n-dimensional space, in such a manner that unit vector $S_1$ is transformed into the new unit vector,

$$S_1U' = (1,0, \ldots ,0). \tag{39}$$

Now apply transformation U to Y by letting

$$V = YU' = (V(1),V(2), \ldots ,V(n)). \tag{40}$$

Then the test function W in (36) reduces to

$$W = V(1)'(VV')^{-1}V(1). \tag{41}$$

For more details on how U acts upon signal $S_1$ and Z or Y under $H_1$, see Appendix.

The covariance matrix of V(i), for i=1,2, ... ,n, is similar to that of Y(i), the only change of the statistics of the V(i) from that of the Y(i) is their mean values under hypothesis $H_1$. This mean is derived from

$$E(V;H_1) = E(YU';H_1)$$
$$= Q^{-1/2}bS_1U'(SS')^{1/2} =$$

$$= Q^{-1/2} b(1,0,\ \ldots\ ,0)(SS')^{1/2}$$

$$= (Q^{-1/2} b(SS')^{1/2},0,\ \ldots\ ,0). \tag{42}$$

From (29) and (42) a figure of merit or what might be termed the generalized signal-to-noise ratio (GSNR) is developed as follows:

$$\text{GSNR} = E(V'(1);H_1)E(V(1);H_1)$$

$$= (b'Q^{-1}b)\|S\|^2 \triangleq a. \tag{43}$$

Now consider a further simplification of the test function W in (41). First separate matrix V into two parts, $V=(V(1),X)$ and $X=(V(2),\ \ldots\ ,V(n))$, in such a manner that

$$VV' = V(1)V'(1) + \sum_{i=2}^{n} V(i)V'(i)$$

$$= V(1)V'(1) + D. \tag{44}$$

Here

$$D = XX' = \sum_{i=2}^{n} V(i)V'(i) \tag{45}$$

is a nonsingular p x p matrix, since $n-1 \geq p$ and D is obviously Wishart distributed (see Appendix for more, but similar, details).

A well-known matrix inversion identity applied to (44) produces the result

$$(VV')^{-1} = (V(1)V'(1) + D)^{-1}$$

$$= \left[ I - \frac{D^{-1}V(1)V'(1)}{1 + V'(1)D^{-1}V(1)} \right] D^{-1}. \tag{46}$$

A substitution of (46) into the test function W in (41) yields the test function W as the new expression

$$W = \frac{V'(1)D^{-1}V(1)}{1 + V'(1)D^{-1}V(1)} = \frac{W_1}{1 + W_1} \tag{47}$$

where

$$W_1 = V'(1)D^{-1}V(1). \tag{48}$$

It is desired now to find the probability density, $f(w_1;H_1)$, of $W_1$ in (48). First reexpress (48) in the form

$$W_1 = \|V(1)\|^2 \left[ \frac{V'(1)}{\|V(1)\|} (XX')^{-1} \frac{V(1)}{\|V(1)\|} \right] . \tag{49}$$

Then normalize the p-component vector V(1) as follows:

$$A(1) = \frac{V(1)}{\|V(1)\|} . \tag{50}$$

Hence by (50) one obtains $W_1$ in (49) in the form

$$W_1 = \|V(1)\|^2 (A'(1)(XX')^{-1}A(1)) = \|V(1)\|^2 q \tag{51}$$

where

$$q = A'(1)(XX')^{-1}A(1). \tag{52}$$

Now one can further process the term q in (52) by conditioning on the elements of V(1) so that A(1) can be treated as a normalized constant vector. Then since A(1) has unity magnitude, there exists a p x p orthonormal matrix $U_1$ such that (see also Muirhead (1982))

$$U_1 A(1) = (1,0, \ldots ,0)'. \tag{53}$$

Next apply this transformation to matrix X, defined before (44), by letting

$$H = U_1 X = U_1(V(2), \ldots ,V(n)). \tag{54}$$

Then the term q in (52) has the simple form,

$$\begin{aligned} q &= A'(1)(XX')^{-1}A(1) \\ &= (1,0, \ldots ,0)(HH')^{-1}(1,0, \ldots ,0)'. \end{aligned} \tag{55}$$

Clearly H in (54) has exactly the same statistical properties as X under the assumption that V(1) is given.

Now partition H as follows:

$$H = \begin{pmatrix} H_A' \\ H_B \end{pmatrix} \tag{56}$$

where $H_A$ is the (n-1)-column vector and $H_B$ is the (p-1) x (n-1) matrix. Then

$$(HH')^{-1} = \begin{pmatrix} H_A'H_A & H_A'H_B' \\ H_BH_A & H_BH_B' \end{pmatrix}^{-1} = \begin{pmatrix} H_{AA} & H_{AB} \\ H_{BA} & H_{BB} \end{pmatrix} . \tag{57}$$

According to the Frobenius relations (Muirhead, 1982) for a partitioned matrix

$$H_{AA} = (H_A'H_A - H_A'H_B'(H_BH_B')^{-1}H_BH_A)^{-1}$$

$$= (H_A'(I - H_B'(H_BH_B')^{-1}H_B)H_A)^{-1}$$

$$= \frac{1}{H_A'(I - H_B'(H_BH_B')^{-1}H_B)H_A} = \frac{1}{H_A'RH_A} . \tag{58}$$

A substitution of (57) and (58) into (55) yields

$$g = \frac{1}{H_A'RH_A} \tag{59}$$

where $R=I-H_B'(H_BH_B')^{-1}H_B$ is a projection operator such that $R^2=R$ and $Tr(R)=n-p$. It is not difficult to show that R has n-p unity eigenvalues and p-1 zero eigenvalues. Thus R can be diagonalized to the form

$$U_2'RU_2 = T = \begin{pmatrix} I_{n-p} & 0 \\ 0 & 0_{p-1} \end{pmatrix} . \tag{60}$$

Under the assumption that V(1) and R are given, one finds also that the random variable

$$1/g = H_A'RH_A = h'h = \sum_{i=1}^{n-p} h_i^2 \tag{61}$$

where g is defined in (59), and

$$h \triangleq T^{1/2}U_2'H_A \tag{62}$$

is a (n-1)-column vector with the last (p-1) components equal to zero. The conditional joint probability density function of the first (n-p) nonzero elements of h is subject to the normal density function, $N(0,I_{n-p})$, i.e.,

$$f(h_1, \dots ,h_{n-p};V(1),R) = N(0,I_{n-p}). \tag{63}$$

But also $f(\cdot;\cdot,\cdot)$ in (63) does not depend on V(1) and R, so that h and 1/g in (61) must be statistically independent of the vector V(1) and matrix R. Hence 1/g in (61) is chi-squared distributed.

Also by (51) and (59) one obtains the ratio $W_1$ in the form

$$W_1 = \frac{V'(1)V(1)}{h'h} = \sum_{j=1}^{p} V_j^2(1) \Big/ \sum_{i=1}^{n-p} h_i^2 . \tag{64}$$

Using the independence of vectors h and V(1), one has the probability density function (see Miller (1964))

$$f(w_1;H_1) = e^{-\grave{a}/2}\,\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-p}{2})\Gamma(\frac{p}{2})}\,\frac{w_1^{(p/2)-1}}{(1+w_1)^{n/2}}\;{}_1F_1\!\left(\frac{n}{2},\frac{p}{2},\frac{\grave{a}}{2}\,\frac{w_1}{1+w_1}\right)$$

$$= f(w_1;p,n-p,\grave{a}^{1/2}) \qquad (65)$$

of $W_1$ in (48) under hypothesis $H_1$ for $0 < w_1 < \infty$ , where ${}_1F_1(a'',b'',x)$ is the confluent hypergeometric function. This is a noncentral f-distribution (Rao, 1965). The value $\grave{a}$ in Miller (1964) is given by

$$\grave{a} = \left|\left|E(V(1);H_1)\right|\right|^2. \qquad (66)$$

This agrees with the definition of the GSNR, $\grave{a}$, given in (43).

Finally by using the relationship of $W_1$ to $W$ in (47) the probability density function of the test function $W$ under hypothesis $H_1$ is given by

$$f(w;H_1) = e^{-\grave{a}/2}\,B(w;\tfrac{p}{2},\tfrac{n-p}{2})\;{}_1F_1(\tfrac{n}{2},\tfrac{p}{2},\tfrac{\grave{a}w}{2}) = B(w;\tfrac{p}{2},\tfrac{n-p}{2},\grave{a}^{1/2}) \qquad (67)$$

for $0 < w < 1$. This says that $W$ is subject to a noncentral beta-distribution. Clearly, if no signal is present, then $\grave{a}=0$. Thus (67) reduces in the $H_0$ hypothesis to a standard beta-function density of form

$$f(w;H_0) = B(\tfrac{p}{2},\tfrac{n-p}{2})^{-1}\,w^{(p-2)/2}\,(1-w)^{(n-p-2)/2},\quad 0 < w < 1. \qquad (68)$$

Finally, in terms of the above probability density functions in (67) and (68) the probability of a false alarm is found by

$$P_{FA} = \int_{w_0}^{1} f(w;H_0)\,dw \qquad (69)$$

and the probability of detection by

$$P_D = \int_{w_0}^{1} f(w;H_1)\,dw. \qquad (70)$$

## APPENDIX

### Properties of Orthogonal Matrix U

To see explicitly how orthonormal transformation $U$ in (37) acts upon the normalized version $S_1$ of signal $S$ in (3) and data matrix $Z$ under $H_1$, the signal-plus-noise situation, let

$$Z \triangleq b_1 S_1 + Z^\circ \tag{A1}$$

where $b_1 = b(SS')^{1/2}$, $S_1 = S(SS')^{-1/2}$, and $Z^\circ$ is data matrix Z when $b_1 = 0$ or hypothesis $H_0$ is true. Since

$$S_1 U' = S_1(S_1', M') = (S_1 S_1', S_1 M') = (1, 0, \ldots, 0), \tag{A2}$$

then a multiplication of Z in (A1) on the right by $U'$ yields

$$V_1 = ZU' = (b_1 S_1 + Z^\circ)\,(S_1', M')$$

$$= ((b_1 S_1 + Z^\circ) S_1', (b_1 S_1 + Z^\circ) M')$$

$$= (b_1 + Z^\circ S_1', Z^\circ M'). \tag{A3}$$

Evidently the action of U on Z is to send signal $b_1 S_1$ to $b_1(1, 0, \ldots, 0)$ plus a noise term $Z^\circ S_1$ into the first column only of the transformed data matrix $V_1$. The remaining n-1 columns of $V_1$, namely $Z^\circ M'$, constitute a signal-free p x (n-1) matrix from which, as it is shown next, an estimator of the covariance matrix Q can be found.

First note by the definition of $V_1$ and U that

$$V_1 V_1' = Z(S_1', M')\,(S_1', M')' Z'$$

$$= (ZS_1')(ZS_1')' + (ZM')(ZM')'. \tag{A4}$$

Finally under $H_1$ by (A3) one has

$$V_1 V_1' = ((b_1 S_1 + Z^\circ) S_1', Z^\circ M')\,((b_1 S_1 + Z^\circ) S_1', Z^\circ M')'$$

$$= ((b_1 S_1 + Z^\circ) S_1')\,((b_1 S_1 + Z^\circ) S_1')'$$

$$+ (Z^\circ M')(Z^\circ M')'$$

$$= (ZS_1')(ZS_1')' + (Z^\circ M')(Z^\circ M')'. \tag{A5}$$

The above identities (A4) to (A5) evidently establish the following relations:

$$(ZM')(ZM')' = (Z^\circ M')(Z^\circ M')'$$

$$= ZZ' - (ZS_1')(ZS_1')' \triangleq G. \tag{A6}$$

Taking the expected value of the left side of (A6) yields, by the use of the independence of the columns of Z,

$$E((ZM)(ZM)') = E((Z^\circ M)(Z^\circ M)') = (n-1)Q \tag{A7}$$

where Q is p x p covariance matrix of the discrete vector process Z(i). This result shows that

$$\hat{Q} = \frac{1}{n-1}(ZM)(ZM)' \tag{A8}$$

is an unbiased estimator of Q with (n-1) degrees of freedom under both of the noise-only and signal-plus-noise hypotheses $H_0$ and $H_1$, respectively. Note finally by (Cramer, 1958) if $n-1 \geq p$ or $n > p$ that the $r=(1/2)p(p+1)$, distinct elements of $\hat{Q}$ are jointly Wishart distributed over the r-dimensional space of positive definite matrices. Hence $\hat{Q}$ in (A8) is positive definite with probability one.

The inverse of ZZ′ can be shown explicitly to likewise exist by noting the following:

$$ZZ' = G + (ZS_1')(ZS_1')' = (I + (ZS_1')(ZS_1')'G^{-1})G \triangleq KG. \tag{A9}$$

Evidently ZZ′ is invertible since clearly both factors $K^{-1}$ and $G^{-1}$ exist with probability one.

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